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# Lower bounding the boundary of a graph in terms of its maximum or minimum degree

Tobias Müller\*    Attila Pór<sup>†‡</sup>    Jean-Sébastien Sereni<sup>†§</sup>

## Abstract

A vertex  $v$  of a graph  $G$  is a *boundary vertex* if there exists a vertex  $u$  such that the distance in  $G$  from  $u$  to  $v$  is at least the distance from  $u$  to any neighbour of  $v$ . We give the best possible lower bound, up to a constant factor, on the number of boundary vertices of a graph in terms of its minimum degree (or maximum degree). This settles a problem introduced by Hasegawa and Saito.

## 1 Introduction

Let  $G = (V, E)$  be a finite, simple and connected graph. For every vertex  $v \in V$ , let  $N(v) := \{u \in V : uv \in E\}$  be the *neighbourhood* of  $v$ . A vertex  $v \in V$  is a *boundary vertex* of  $G$  if there exists a vertex  $u \in V$  such that  $\text{dist}(u, v) \geq \text{dist}(u, w)$  for every  $w \in N(v)$ . Such a vertex  $u$  is a *witness* for  $v$ . The *boundary* of  $G$  is the set  $\mathcal{B}(G)$  of boundary vertices of  $G$ .

The notion of boundary was introduced by Chartrand *et al.* [2, 3] and further studied by Cáceres *et al.* [1], Hernando *et al.* [5], and Hasegawa and Saito [4]. The latter proved that for every graph  $G$ ,

$$\delta(G) \leq r \left( \binom{|\mathcal{B}(G)| - 1}{2} * 4 \right), \quad (1)$$

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where  $\delta(G)$  is the minimum degree of  $G$ . The right-hand side of (1) is the (multicoloured) Ramsey number  $r\left(\binom{|\mathcal{B}(G)|-1}{2} * 4\right)$ , that is the smallest integer  $n$  such that each colouring of the edges of  $K_n$  with  $\binom{|\mathcal{B}(G)|-1}{2}$  colours yields a monochromatic copy of  $K_4$ . As shown by Xiaodong *et al.* [6], the Ramsey number  $r(k * 4)$  is  $\Omega(5^k)$ . Therefore, the lower bound on the number of boundary vertices yielded by (1) cannot be better than

$$|\mathcal{B}(G)| = \Omega\left(\sqrt{\log(\delta(G))}\right).$$

The following result is a significant improvement.

**Theorem 1.** *For every graph  $G$  of maximum degree  $\Delta$ ,*

$$|\mathcal{B}(G)| \geq \log_2(\Delta + 2).$$

The bound provided by Theorem 1 is sharp up to a multiplicative factor smaller than  $3 \log_3(2) \approx 1.89$ .

**Theorem 2.** *For every positive integer  $n$ , there exists a graph  $G_n$  of minimum degree  $\delta_n := 3^{n-1}$  and maximum degree  $\Delta_n := 3^n + n - 1$  with  $|\mathcal{B}(G_n)| = 3n$ . Thus,  $|\mathcal{B}(G_n)| = 3(\log_3(\delta_n) + 1) < 3 \log_3(\Delta_n)$ .*

In particular, we deduce that the lower bound on the size of the boundary in terms of the *minimum* degree implied by Theorem 1 is essentially best possible, which answers a question of Hasegawa and Saito [4].

As it happens the vertex-connectivity of the graph  $G_n$  in Theorem 2 is also  $3^{n-1}$ , which shows that being highly vertex-connected is not a sufficient condition for having a large boundary.

## 2 Upper bound on the maximum degree

Throughout this section, let  $G = (V, E)$  be a graph. The *endvertices* of a path  $P$  of  $G$  are the two vertices of degree 1 in  $P$ . A *shortest path* of  $G$  is a path whose length is precisely the distance in  $G$  between its endvertices. Given a shortest path  $P$ , an *extension* of  $P$  is a shortest path  $Q$  containing  $P$ . If  $Q$  is an extension of  $P$ , we say that  $P$  *extends* to  $Q$ . The proof of Theorem 1 relies on the following observation.

**Lemma 3.** *Each shortest path of  $G$  extends to a shortest path between two boundary vertices.*

*Proof.* Let  $P$  be a shortest path of  $G$ . If one of its endvertices is not a boundary vertex, then  $P$  extends to a longer path (which is also a shortest path between its endvertices), by the definition of a boundary vertex. As the graph  $G$  is finite, we eventually obtain an extension of  $P$  whose endvertices are boundary vertices.  $\square$

For every vertex  $v \in V$ , let  $C_v : N(v) \rightarrow 2^{\mathcal{B}(G)}$  be the mapping defined by

$$C_v(u) := \{b \in \mathcal{B}(G) : \text{dist}(b, u) < \text{dist}(b, v)\}.$$

The proof of the following lemma relies on Lemma 3.

**Lemma 4.** *Let  $v \in V$ . For each pair  $(u, u')$  of neighbours of  $v$ ,  $C_v(u) \neq C_v(u')$ . Moreover,  $C_v(u)$  is neither empty nor the whole set  $\mathcal{B}(G)$ .*

*Proof.* By Lemma 3 there exists a path  $P$  containing the vertices  $u$  and  $u'$  that is a shortest path between two boundary vertices  $b$  and  $b'$ . We may assume that  $u$  is closer to  $b$  than  $u'$ . Let  $r := \text{dist}(u, b)$  and  $s := \text{dist}(u', b')$ .

First suppose that  $uu' \notin E$ . In this case we may assume that  $v$  belongs to  $P$ . Since  $P$  is a shortest path,  $\text{dist}(v, b) = r+1 = \text{dist}(u', b)-1$ . Consequently,  $b \in C_v(u) \setminus C_v(u')$ .

Assume now that  $uu' \in E$ . Since  $\text{dist}(v, b) + \text{dist}(v, b') \geq \text{dist}(b, b') = r + s + 1$ , it follows that  $\text{dist}(v, b) \geq r + 1$  or  $\text{dist}(v, b') \geq s + 1$ . By symmetry, assume that  $\text{dist}(v, b) = r + 1$ . Since  $P$  is a shortest path between  $b$  and  $b'$ , we deduce that  $\text{dist}(u', b) = r + 1$ , and therefore  $b \in C_v(u) \setminus C_v(u')$ .

Since the edge  $uv$  extends to a shortest path between two boundary vertices, we infer that  $C_v(u)$  is neither empty nor the whole set  $\mathcal{B}(G)$ , which concludes the proof.  $\square$

*Proof of Theorem 1.* Let  $v$  be a vertex of  $G$  of degree at least 2. By Lemma 4, there exists an injective mapping from  $N(v)$  to  $2^{\mathcal{B}(G)} \setminus \{\emptyset, \mathcal{B}(G)\}$ . Therefore the degree of  $v$  is at most  $2^{|\mathcal{B}(G)|} - 2$ , which yields the desired result.  $\square$

### 3 Construction of the graph $G_n$

Fix a positive integer  $n$ . Let the vertex-set of the graph  $G_n$  be  $V := A \cup B$  where

$$A := \{0, 1, 2\}^n, \quad B := \{b_j^i : j \in \{1, \dots, n\} \text{ and } i \in \{0, 1, 2\}\}.$$

Let the edge-set of the graph  $G_n$  be

$$E := \{uv : u, v \in A\} \cup \bigcup_{\substack{j \in \{1, \dots, n\} \\ i \in \{0, 1, 2\}}} \{vb_j^i : v \in A \text{ and } (v)_j = i\}.$$

The vertex  $b_j^i$  is joined to exactly those vertices  $v \in A$  whose  $j$ -th coordinate is  $i$ . Notice that the vertices of  $A$  have degree  $3^n + n - 1$ , and those of  $B$  have degree  $3^{n-1}$ . So it only remains to establish that  $\mathcal{B}(G_n) = B$ .

Note that the diameter of  $G$  is 3. For every two indices  $i \neq i'$ , and every  $j \in \{1, 2, \dots, n\}$ , the path  $b_j^i v w b_j^{i'}$  is a shortest path of length 3, where  $v, w \in A$  with  $(v)_j = i$  and  $(w)_j = i'$ . Since every pair of vertices that are not both in  $B$  lie on such a shortest path, it follows that  $\mathcal{B}(G) = B$ .

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